Differential Geometry III: Riemannian geometry

1.1 Let $\mathcal{M}$ be a differentiable manifold of dimension m. Assume that $\mathcal{U}, \mathcal{U}'$ are two open subsets of $\mathcal{M}$ with $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, equipped with coordinate charts $\phi : \mathcal{U} \to \mathcal{V} \subset \mathbb{R}^m$ and $\phi' : \mathcal{U}' \to \mathcal{V}' \subset \mathbb{R}^m$. Let $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^m)$ be the corresponding coordinates on $\mathcal{U}$ and $\mathcal{U}'$, respectively; recall that each coordinate function $x^i : \mathcal{U} \to \mathbb{R}$ is defined so that

$$x^i = \bar{x}^i \circ \phi,$$

where $\bar{x}^i : \mathbb{R}^m \to \mathbb{R}$ is the projection on the i-th coordinate (equivalently, for any $q \in \mathcal{U}$, $x^i(q)$ is equal to the i-th component of the vector $\phi(q) \in \mathbb{R}^m$); similarly for y^i (with ϕ' in place of ϕ).

(a) Prove that the functions

$$\tilde{y}^i = y^i \circ \phi^{-1}, \quad i = 1, \ldots, m$$

are differentiable functions on $\phi(\mathcal{U} \cap \mathcal{U}') \subset \mathbb{R}^m$ (*Hint: Use the assumption on the smoothness of transition functions on* $\mathcal{M}$). Show also that

$$\tilde{y}^i(x^1(p), \ldots, x^m(p)) = y^i(p) \quad \text{for all } p \in \mathcal{U} \cap \mathcal{U}'$$

Remark. We usually refer to the function $\tilde{y}^i$ *as the expression of the coordinate function* y^i *with respect to the* $(x^1, \ldots, x^m)$ *coordinate system on* $\mathcal{U} \cap \mathcal{U}'$.

(b) Show that the coordinate tangent vectors $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^m$ and $\left\{\frac{\partial}{\partial y^i}\right\}_{i=1}^m$ satisfy at every point on $\mathcal{U} \cap \mathcal{U}''$

$$\frac{\partial}{\partial x^i} = \partial_i \tilde{y}^j \circ \phi \cdot \frac{\partial}{\partial y^j}.$$

Find a similar relation between the coordinate covectors $\{dx^i\}_{i=1}^m$ and $\{dy^i\}_{i=1}^m$.

Solution. (a) The definition of the coordinate functions $y^i : \mathcal{U}' \to \mathbb{R}$, $1 \leqslant i \leqslant m$, implies that

$$\tilde{y}^i = y^i \circ \phi^{-1} = \bar{x}^i \circ (\phi' \circ \phi^{-1}). \tag{1}$$

Our assumption that $\mathcal{M}$ is a differentiable manifold implies that the transition map

$$\phi' \circ \phi^{-1} : \phi(\mathcal{U} \cap \mathcal{U}') \subset \mathbb{R}^m \to \phi'(\mathcal{U} \cap \mathcal{U}') \subset \mathbb{R}^m$$

is a C^∞ homeomorphism between two open subsets of $\mathbb{R}^m$. Moreover, the coordinate projection map $\bar{x}^i : \mathbb{R}^m \to \mathbb{R}$ is a C^∞ map. Thus, the composition of the function defining $\tilde{y}^i$ in (1) must be a C^∞ function.

The fact that $\bar{x}^i : \mathbb{R}^m \to \mathbb{R}$ is the projection to the i-th Cartesian coordinate is equivalent to the statement that, for any point $z \in \mathbb{R}^m$,

$$z = (\bar{x}^1(z), \ldots, \bar{x}^m(z)).$$

Thus, for any $p \in \mathcal{U} \cap \mathcal{U}'$,

$$\phi(p) = (\bar{x}^1 \circ \phi(p), \ldots, \bar{x}^m \circ \phi(p)) = (x^1(p), \ldots, x^m(p))$$

and, therefore,

$$y^i(p) = \tilde{y}^i \circ \phi(p) = \tilde{y}(x^1(p), \ldots, x^m(p))$$

(b) Recall that, in any local system of coordinates $(y^1, \ldots, y^m)$, the coordinate vector fields $\frac{\partial}{\partial y^i}$ are defined so that the result of their action on the coordinate functions y^j is

$$\frac{\partial}{\partial y^i}(y^j) = \delta^j_i, \quad i, j = 1, \ldots, m. \tag{2}$$

At any point $p \in \mathcal{U} \cap \mathcal{U}'$, both sets of tangent vectors $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^m$ and $\left\{\frac{\partial}{\partial y^i}\right\}_{i=1}^m$ constitute a basis for $T_p\mathcal{M}$; as a result, there exist functions $\lambda^j_i : \mathcal{U} \cap \mathcal{U}' \to \mathbb{R}$, $i, j = 1, \ldots, m$, such that, for any $i = 1, \ldots, m$:

$$\frac{\partial}{\partial x^i} = \lambda^j_i \frac{\partial}{\partial x^j} \tag{3}$$

(recall that repeated indices are assumed to be summed). Using (3) to compute $\frac{\partial}{\partial x^i}(y^k)$, together with (2), we therefore obtain

$$\frac{\partial}{\partial x^i}(y^k) = \lambda^j_i \frac{\partial}{\partial y^j}(y^k) = \lambda^k_i. \tag{4}$$

Using the expression

$$\tilde{y}^k(x^1(\cdot), \ldots, x^m(\cdot)) = y^k(\cdot),$$

we can also calculate (after applying the chain rule) that

$$\frac{\partial}{\partial x^i}(y^k)(p) = \partial_i \tilde{y}^k \circ \phi(p). \tag{5}$$

Thus, returning to (3) and using (4)–(5), we obtain

$$\frac{\partial}{\partial x^i} = \partial_i \tilde{y}^j \circ \phi \cdot \frac{\partial}{\partial y^j}. \tag{6}$$

Similarly, using the fact that both $\{dx^i\}_{i=1}^m$ and $\{dy^i\}_{i=1}^m$ form a basis of $T^*_p\mathcal{M}$ for any $p \in \mathcal{U} \cap \mathcal{U}'$, we have

$$dy^i = f^i_j dx^j \tag{7}$$

for some functions $f^i_j : \mathcal{U} \cap \mathcal{U}' \to \mathbb{R}$, $i, j = 1, \ldots, m$. We can therefore compute

$$\frac{\partial}{\partial x^j}(y^i) = dy^i\big(\frac{\partial}{\partial x^j}\big) = f^i_k dx^k\big(\frac{\partial}{\partial x^j}\big) = f^i_k \delta^k_j = f^i_j.$$

However, using (6), we also have the following expression for $\frac{\partial}{\partial x^j}(y^i)$:

$$\frac{\partial}{\partial x^j}(y^i) = \partial_j \tilde{y}^k \circ \phi \cdot \frac{\partial}{\partial y^k}(y^i) = \partial_j \tilde{y}^k \circ \phi \cdot \delta^k_i = \partial_j \tilde{y}^i \circ \phi.$$

Therefore, combining the above two relations and returning to (7), we infer:

$$dy^i = \big(\partial_j \tilde{y}^i \circ \phi\big) \cdot dx^j.$$

1.2 Construct a smooth atlas (not necessarily maximal) on the unit sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

Solution. An example of an atlas on $\mathbb{S}^n$ consists of the collection $\{(\mathcal{U}_\pm^{(k)}, \phi_\pm^{(k)})\}_{k=1}^{n+1}$ where:

– The open sets $\mathcal{U}_+^{(k)}, \mathcal{U}_-^{(k)} \subset \mathbb{S}^n$, $k = 1, \dots, n+1$ defined by

$$\mathcal{U}_+^{(k)} = \big\{x = (x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : \|x\| = 1 \text{ and } x^k > 0\big\},$$
$$\mathcal{U}_-^{(k)} = \big\{x = (x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : \|x\| = 1 \text{ and } x^k < 0\big\}$$

(note that every point $x \in \mathbb{S}^n$ has at least one non-zero coordinate and thus belongs to at least one of the sets $\mathcal{U}_\pm^{(k)}$). Note that $\mathcal{U}_+^{(k)}$ and $\mathcal{U}_-^{(k)}$ correspond, respectively, to the north and south hemispheres of $\mathbb{S}^n$ in the direction of the x^k-axis (i.e. with the equator corresponding to $\mathbb{S}^n \cap \{x^k = 0\}$).

– The homeomorphisms $\phi_\pm^{(k)} : \mathcal{U}_\pm^{(k)} \to B_1^n \subset \mathbb{R}^n$ (where $B_1^n = \{y \in \mathbb{R}^n : \|y\| < 1\}$) are defined as the projections on the coordinate hyperplane orthogonal to the x^k-axis, i.e.

$$\phi_\pm^{(k)}(x^1, \dots, x^{k-1}, x^k, x^{k+1}, \dots x^{n+1}) = (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^{n+1}).$$

(it is easy to check that they are continuous, $1-1$ and onto). Note that the inverse maps $(\phi_\pm^{(k)})^{-1} : B_1^n \to \mathcal{U}_\pm^{(k)}$ take the form

$$(\phi_+^{(k)})^{-1}(y^1, \dots, y^n) = (y^1, \dots, y^{k-1}, +\sqrt{1 - \sum_{i=1}^n (y^i)^2}, y^k, \dots, y^n), \tag{8}$$

$$(\phi_-^{(k)})^{-1}(y^1, \dots, y^n) = (y^1, \dots, y^{k-1}, -\sqrt{1 - \sum_{i=1}^n (y^i)^2}, y^k, \dots, y^n). \tag{9}$$

For any numbers $k_1, k_2 \in \{1, \dots, n+1\}$ and any signs $\epsilon_1, \epsilon_2 \in \{+, -\}$, the image of the set $\mathcal{U}_{\epsilon_1}^{(k_1)} \cap \mathcal{U}_{\epsilon_2}^{(k_2)}$ via the chart $\phi_{\epsilon_1}^{(k_1)}$ satisfies

$$\Omega_{\epsilon_1,\epsilon_2}^{(k_1,k_2)} \doteq \phi_{\epsilon_1}^{(k_1)}\big(\mathcal{U}_{\epsilon_1}^{(k_1} \cap \mathcal{U}_{\epsilon_2}^{(k_2)}\big) = \begin{cases} B_1^n \cap \{x^{k_2} > 0\} & \text{if } k_1 > k_2 \text{ and } \epsilon_2 = +, \\ B_1^n \cap \{x^{k_2} < 0\} & \text{if } k_1 > k_2 \text{ and } \epsilon_2 = -, \\ B_1^n \cap \{x^{k_2-1} > 0\} & \text{if } k_1 < k_2 \text{ and } \epsilon_2 = +, \\ B_1^n \cap \{x^{k_2-1} < 0\} & \text{if } k_1 < k_2 \text{ and } \epsilon_2 = -, \\ B_1^n & \text{if } k_1 = k_2 \text{ and } \epsilon_1 = \epsilon_2, \\ \emptyset & \text{if } k_1 = k_2 \text{ and } \epsilon_1 \neq \epsilon_2. \end{cases}$$

In view of the above explicit formulas for $\phi_\pm^{(k)}$ and $(\phi_\pm^{(k)})^{-1}$, it is straightforward to verify that the transition maps $\phi_{\epsilon_2}^{(k_2)} \circ (\phi_{\epsilon_1}^{(k_1)})^{-1}$ are smooth (in fact, real analytic) homeomorphisms from $\Omega_{\epsilon_1,\epsilon_2}^{(k_1,k_2)} \subset \mathbb{R}^n$ to $\Omega_{\epsilon_1,\epsilon_2}^{(k_2,k_1)} \subset \mathbb{R}^n$. Thus, $\{(\mathcal{U}_\pm^{(k)}, \phi_\pm^{(k)})\}_{k=1}^{n+1}$ is a smooth atlas on $\mathbb{S}^n$ (albeit not maximal).

1.3 Let $(\mathcal{M}, g)$ be a Riemannian manifold and let $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ be two systems of coordinates around a point $p \in \mathcal{M}$. Let g_{ij} be the components of the metric g in the $(x^1, \ldots, x^n)$ coordinates, while $\tilde{g}_{ij}$ are the components of g with respect to $(y^1, \ldots, y^n)$. Show that

$$\tilde{g}_{ij} = \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j} g_{ab},$$

where, in the above expressions, the coordinate functions x^γ are considered as functions of $(y^1, \ldots, y^n)$ (see Ex. 1.1). Express the Euclidean metric on $\mathbb{R}^2 \setminus \{0\}$ in polar coordinates.

Solution. Using the relations from Ex. 1.1 between the coordinate tangent vectors $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ and $\left\{\frac{\partial}{\partial y^i}\right\}_{i=1}^n$, we can easily calculate:

$$\tilde{g}_{ij} = g\big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\big) = g\big(\frac{\partial x^a}{\partial y^i}\cdot\frac{\partial}{\partial x^a}, \frac{\partial x^b}{\partial y^j}\cdot\frac{\partial}{\partial x^b}\big) = \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j} g\big(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\big) = \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j} g_{ab}.$$

Note that we made use of the fact that, at any $p \in \mathcal{M}$, $g|_p$ is bilinear on $T_p\mathcal{M} \times T_p\mathcal{M}$ and therefore, for any finite set of vector fields $\{X_\alpha\}_\alpha, \{Y_\beta\}_\beta$ and functions $\{f^{(1)}_\alpha\}_\alpha, \{f^{(2)}_\beta\}_\beta : \mathcal{M} \to \mathbb{R}$, we have:

$$g\big(\sum_\alpha f^{(1)}_\alpha X_\alpha, \sum_\beta f^{(2)}_\beta Y_\beta\big) = \sum_\alpha\sum_\beta f^{(1)}_\alpha f^{(2)}_\beta g(X_\alpha, Y_\beta).$$

Another way to obtain the same identity is by noting that g_{ij} and $\tilde{g}_{ij}$ are the components of g with respect to the coordinate bases $\{dx^i \otimes dx^j\}_{i,j=1}^n$ and $\{dy^i \otimes dy^j\}_{i,j=1}^n$, respectively (these are the coordinate bases of bilinear functionals on $T_p\mathcal{M} \times T_p\mathcal{M}$ associated to each of the coordinate systems $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$). Therefore, we have

$$g = g_{ab} dx^a \otimes dx^b = \tilde{g}_{ij} dy^i \otimes dy^j.$$

Using the relations from Ex. 1.1 between the coordinate covectors dx^a and dy^j, we can also calculate:

$$g = g_{ab} dx^a \otimes dx^b = g_{ab}\frac{\partial x^a}{\partial y^i} dy^i \otimes \frac{\partial x^b}{\partial y^j} dy^j = \Big(g_{ab}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\Big) dy^i \otimes dy^j$$

The above two relations now imply (since $\{dy^i \otimes dy^j\}_{i,j=1}^n$ forms a basis of $T^*_p\mathcal{M} \otimes T^*_p\mathcal{M}$ at any $p \in \mathcal{M}$) that

$$\tilde{g}_{ij} = g_{ab}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}.$$

The polar coordinates (r, θ) on $\mathbb{R}^2 \setminus 0$ are related to the Cartesian coordinates (x^1, x^2) as follows:

$$x^1 = r\cos\theta, \quad x^2 = r\sin\theta.$$

Thus, we can compute:

$$dx^1 = \cos\theta dr - r\sin\theta d\theta, \quad dx^2 = \sin\theta dr + r\cos\theta d\theta.$$

The Euclidean metric g_E therefore takes the form

$$\begin{aligned}
g_E &= dx^1 \otimes dx^1 + dx^2 \otimes dx^2 = \\
&= (\cos\theta dr - r\sin\theta d\theta) \otimes (\cos\theta dr - r\sin\theta d\theta) + (\sin\theta dr + r\cos\theta d\theta) \otimes (\sin\theta dr + r\cos\theta d\theta) \\
&= \cos^2\theta(dr \otimes dr) - r\sin\theta\cos\theta(dr \otimes d\theta + d\theta \otimes dr) + r^2\sin^2\theta(d\theta \otimes d\theta) \\
&\qquad + \sin^2\theta(dr \otimes dr) + r\sin\theta\cos\theta(dr \otimes d\theta + d\theta \otimes dr) + r^2\cos^2\theta(d\theta \otimes d\theta) \\
&= dr \otimes dr + r^2 d\theta \otimes d\theta
\end{aligned}$$

1.4 Let $\mathcal{M}$ be a differentiable manifold and $F : \mathcal{M} \to \mathbb{R}^N$ be an immersion. The metric g *induced* on $\mathcal{M}$ by the Euclidean metric on $\mathbb{R}^N$ is defined by the relation

$$g(X, Y) \doteq \langle dF(X), dF(Y)\rangle_{\mathbb{R}^N} \quad \text{for all tangent vectors} X, Y \text{ on } \mathcal{M}.$$

(a) Show that, in any local coordinate system $(x^1, \dots, x^n)$ on $\mathcal{M}$, the components of g are given by

$$g_{ij} = \delta_{ab}\frac{\partial F^a}{\partial x^i}\frac{\partial F^b}{\partial x^j}.$$

(b) **(Surface of revolution)** Let $\gamma : (0,1) \to \mathbb{R}^2$ be a smooth curve parametrized with unit speed (i.e. $\big\langle \frac{d\gamma}{du}(u), \frac{d\gamma}{du}(u)\big\rangle = 1$). Let $\gamma(u) = \big(X(u), Y(u)\big)$ be the representation of γ in the standard Cartesian coordinates on $\mathbb{R}^2$ and assume that $X(u) > 0$ for all $u \in (0,1)$. Consider the surface of revolution $\mathcal{S} \subset \mathbb{R}^3$ obtained by rotating the curve γ around the y-axis; this surface is parametrized by $(u, \theta) \in (0,1) \times [0, 2\pi)$ via the map

$$\Psi(u, \theta) = \big(X(u)\cos\theta,\, Y(u),\, X(u)\sin\theta\big).$$

Express the induced metric on $\mathcal{S}$ from the Euclidean metric on $\mathbb{R}^3$ in the (u, θ) coordinates.

Solution. (a) It is straightforward to calculate

$$\begin{aligned}
g_{ij} &= g\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big) \\
&= \big\langle dF\big(\frac{\partial}{\partial x^i}\big), dF\big(\frac{\partial}{\partial x^j}\big)\big\rangle_{\mathbb{R}^N} \\
&= \big\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\big\rangle_{\mathbb{R}^N} \\
&= \delta_{ab}\frac{\partial F^a}{\partial x^i}\frac{\partial F^b}{\partial x^j},
\end{aligned}$$

where $F = (F^1, \dots, F^N)$ is the expression of F in the Cartesian coordinates of $\mathbb{R}^N$.

(b) Let g be the induced metric on S. For the embedding map $\Psi : (0,1) \times [0, 2\pi) \to \mathbb{R}^3$ we can readily calculate:

$$\frac{\partial \Psi}{\partial u}(u, \theta) = \big(\dot{X}(u)\cos\theta,\, \dot{Y}(u),\, \dot{X}(u)\sin\theta\big),$$

$$\frac{\partial \Psi}{\partial \theta}(u,\theta) = \big(- X(u)\sin\theta,\, 0,\, X(u)\cos\theta\big).$$

Thus, using the formula from part (a) for the components of g in the (u,θ) coordinate system, we calculate:

$$\begin{aligned}
g_{uu} &= \delta_{ab}\frac{\partial \Psi^a}{\partial u}\frac{\partial \Psi^b}{\partial u}\\
&= (\dot{X}(u))^2\cos^2\theta + (\dot{Y}(u))^2 + (\dot{X}(u))^2\sin^2\theta\\
&= (\dot{X}(u))^2 + (\dot{Y}(u))^2,\\
g_{u\theta} &= \delta_{ab}\frac{\partial \Psi^a}{\partial u}\frac{\partial \Psi^b}{\partial \theta}\\
&= \dot{X}(u)X(u)\cos\theta\sin\theta + 0 - \dot{X}(u)X(u)\cos\theta\sin\theta\\
&= 0,\\
g_{\theta\theta} &= \delta_{ab}\frac{\partial \Psi^a}{\partial \theta}\frac{\partial \Psi^b}{\partial \theta}\\
&= (X(u))^2\sin^2\theta + 0 + (X(u))^2\cos^2\theta\\
&= (X(u))^2.
\end{aligned}$$

The assumption that γ is parametrized by unit speed translates to the condition that

$$(\dot{X}(u))^2 + (\dot{Y}(u))^2 = 1.$$

Therefore

$$\begin{aligned}
g &= g_{uu} du\otimes du + g_{u\theta}(du\otimes d\theta + d\theta\otimes du) + g_{\theta\theta} d\theta\otimes d\theta\\
&= du\otimes du + \big(X(u)\big)^2 d\theta\otimes d\theta.
\end{aligned}$$

1.5 Let $(\mathcal{M}, g)$ be a Riemannian manifold and $x^1,\dots,x^n$ a system of local coordinates on an open subset $U\subset\mathcal{M}$ associated to a coordinate chart $\phi : U\to\mathbb{R}^n$. Show that the volume

$$\mathrm{Vol}(U) = \int_{\phi(U)} \sqrt{\det(g_{ij})}\ \mathrm{d}x^1\dots\mathrm{d}x^n$$

is independent of the choice of coordinates.

Solution. Let $\phi' : U\to\phi'(U)\subset R^n$ be a (possibly) different coordinate chart, with associated coordinates $(y^1,\dots,y^n)$. Let $\tilde{g}_{ij}$ be the components of g with respect to the y^i coordinates. Our aim is to show that

$$\int_{\phi(U)} \sqrt{\det(g_{ij})}\circ\phi^{-1}\ \mathrm{d}x^1\dots\mathrm{d}x^n = \int_{\phi'(U)} \sqrt{\det(\tilde{g}_{ij})}\circ(\phi')^{-1}\ \mathrm{d}y^1\dots\mathrm{d}y^n.$$

Let us make a few observations:

- The map $Y = \phi' \circ \phi^{-1}$ is a smooth homeomorphism from $\phi(U) \subset \mathbb{R}^n$ to $\phi'(U) \subset \mathbb{R}^n$ (as a transition map for our manifold). Notice that $Y(\bar{x}^1, \ldots, \bar{x}^n) = \big(Y^1(\bar{x}), \ldots, Y^n(\bar{x})\big)$ is simply the expression of the $(y^1, \ldots, y^n)$ coordinates on U as functions of the $(x^1, \ldots, x^n)$ coordinates (see Ex. 1.1).
- Let us denote for a moment by $G : \phi(U) \to \mathbb{R}$ and $\tilde{G} : \phi'(U) \to \mathbb{R}$ the functions $\sqrt{\det(g_{ij})} \circ \phi^{-1}$ and $\sqrt{\det(\tilde{g}_{ij})} \circ (\phi')^{-1}$, respectively. In matrix notation, the formula from Ex. 1.3 (after changing the roles of x and y there) can be reexpressed as

$$[g] = ([dY] \circ \phi)^T \cdot [\tilde{g}] \cdot ([dY] \circ \phi),$$

where

$$[g] = \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \\ g_{n1} & \cdots & g_{nn} \end{bmatrix}, \quad [\tilde{g}] = \begin{bmatrix} \tilde{g}_{11} & \cdots & \tilde{g}_{1n} \\ \vdots & \ddots & \\ \tilde{g}_{n1} & \cdots & \tilde{g}_{nn} \end{bmatrix}$$

and

$$[dY] = \begin{bmatrix} \frac{\partial Y^1}{\partial x^1} & \cdots & \frac{\partial Y^1}{\partial x^n} \\ \vdots & \ddots & \\ \frac{\partial Y^n}{\partial x^1} & \cdots & \frac{\partial Y^n}{\partial x^n} \end{bmatrix}.$$

Therefore:

$$\begin{aligned} \sqrt{\det[g]} \circ \phi^{-1} &= \sqrt{\det\Big([dY]^T \cdot ([\tilde{g}] \circ \phi^{-1}) \cdot [dY]\Big)} \\ &= |\det[dY]| \sqrt{\det[\tilde{g}]} \circ \phi^{-1} \\ &= |\det[dY]| \sqrt{\det[\tilde{g}]} \circ (\phi')^{-1} \circ Y. \end{aligned}$$

- The classical change of variables formula for integrals on domains in $\mathbb{R}^n$ gives us that, for any continuous function $f : \phi(U) \to \mathbb{R}$, its integral transforms under the map $Y : \phi(U) \to \phi'(U)$ by the relation:

$$\int_{\phi(U)} f(x)\, dx^1 \ldots dx^n = \int_{\phi'(U)} f \circ Y^{-1}(y) \frac{1}{|\det[dY]| \circ Y^{-1}(y)} dy^1 \ldots dy^n.$$

Combining the above observations, we obtain:

$$\begin{aligned} \int_{\phi(U)} \sqrt{\det[g]} \circ \phi^{-1}(x)\, dx^1 \ldots dx^n &= \int_{\phi'(U)} \sqrt{\det[g]} \circ \phi^{-1} \circ Y^{-1}(y) \frac{1}{|\det[dY]| \circ Y^{-1}(y)} dy^1 \ldots dy^n \\ &= \int_{\phi'(U)} \sqrt{\det[g]} \circ (\phi')^{-1}(y) \frac{1}{|\det[dY]| \circ Y^{-1}(y)} dy^1 \ldots dy^n \\ &= \int_{\phi'(U)} |\det[dY] \circ Y^{-1}| \sqrt{\det[\tilde{g}]} \circ (\phi')^{-1}(y) \frac{1}{|\det[dY]| \circ Y^{-1}(y)} dy^1 \ldots dy^n \\ &= \int_{\phi'(U)} \sqrt{\det[\tilde{g}]} \circ (\phi')^{-1}(y)\, dy^1 \ldots dy^n. \end{aligned}$$